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Resonant interactions among capillary-gravity waves

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The analysis presented here is a second-order analysis of the resonant interactions among triads of waves with wavelengths in the capillary-gravity and pure capillary ranges. The analysis is not a power series perturbation analysis, and one of the objects is the removal of the secularity which arises through power series perturbations. It is further found that the interactions are energy-conserving to the order considered here. Suitable modifications are made to accommodate the inevitable viscous attenuation associated with these small wavelengths. A start is made toward describing more completely the various spectra of random seas in wave-number and frequency regions where these interactions are dynamically significant.

1. Introduction

Several years ago, Phillips (1960) uncovered a mechanism for non-linear resonant interaction between two or more trains of progressive gravity waves of certain wave-numbers and directions on the surface of deep water. The analysis of this phenomenon was somewhat extended by Longuet-Higgins (1962), but both used essentially the same method, a perturbation analysis, in which the variables of interest in the problem were expressed as power series in some small parameter ϵ being proportional to the wave slope. Through appropriate linearization by collecting the coefficients of the various powers of ϵ, both predict at the third order (order ϵ^3) a linear growth in time of what they call a resonant tertiary wave, and calculate a characteristic growth time which is proportional to the (-2)-power of the geometric mean of the slopes of the primary waves. Phillips in particular demonstrates that this resonance cannot occur at the second order (order ϵ^2) for deep-water gravity waves, because it is impossible to find closed triads of wave-number satisfying his condition for resonance, namely simultaneous solutions of the equations $\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0$ and $\sigma_1 \pm \sigma_2 \pm \sigma_3 = 0$ where $\sigma_i = (gk_i)^{\frac{1}{2}}$, save for trivial configurations in which one of the wave-numbers involved is zero. At third order, the resonance condition is among quadruplets of wave-numbers and frequencies, or $\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0$ and $\sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0$, where again they assume $\sigma_i = (gk_i)^{\frac{1}{2}}$. Finding that there are indeed solutions to these equations, they proceed with detailed analyses at this order, the results being valid for some short time.

In the present work, it is found that if the effects of surface tension are considered as well as those of gravity, then with the wave-number frequency relation written as $\sigma_i = (gk_i + Tk_i^3)^{\frac{1}{2}}$, it is in fact possible to find certain triads of wave-numbers satisfying the resonance condition, as will be seen in the next section. So for waves in the capillary-gravity range and in the pure capillary range, it is possible that resonances can occur at the second order. The ensuing analysis of the resonant interactions is not, however, a power series perturbation method like those of Phillips and Longuet-Higgins; it is a different type of analysis for which the method of ordering the terms is somewhat different from theirs. The similarities lie in the assumptions of irrotational motion, infinite depth, and in the manner in which the non-linear kinematic and dynamic boundary conditions are approximated at the free surface. In both analyses, the boundary conditions are expanded in a Taylor series about the equilibrium level of the undisturbed free surface, but in the present analysis, it is only necessary to retain the first and second terms in these series when the fundamental variables appear by themselves (from linear terms), and the first terms of the series when the variables appear as quadratic terms or products of two linear terms (from the non-linear terms). The various cubic terms and triple products are neglected, the results being the same, of course, as if the boundary conditions were handled by a perturbation scheme, retaining only terms in ϵ and ϵ^2. The fundamental difference between the two methods, however, is in the specification of the form of the free surface. In the perturbation series, Longuet-Higgins (1962) expresses the free surface $\xi(\mathbf{x}, t)$ as

$$\xi = (\alpha\xi_{10} + \beta\xi_{01}) + (\alpha^2\xi_{20} + \alpha\beta\xi_{11} + \beta^2\xi_{02}) + \dots$$

from which the orders are apparent, and the interaction term is clearly identifiable as ξ_{11}, which appears at the second order. The amplitudes of the first-order terms ξ_{10} and ξ_{01} are allowed to remain constant. In this paper, the free surface at any time t is composed of three waves, the wave-numbers and frequencies of which satisfy the resonance condition for triads, and the amplitudes of which are to be considered slowly varying functions of time, or

$$\begin{aligned}\xi(\mathbf{x}, t) = {} & a_1(t) \cos(\mathbf{k}_1 . \mathbf{x} - \sigma_1 t + \epsilon_1) \\ & + a_2(t) \cos(\mathbf{k}_2 . \mathbf{x} - \sigma_2 t + \epsilon_2) \\ & + a_3(t) \cos(\mathbf{k}_3 . \mathbf{x} - \sigma_2 t + \epsilon_3).\end{aligned}$$

In order to identify one of these components as a product of resonance as in the preceding works, it will be supposed that at some time t_0 the amplitude of one of the waves, say a_3, is zero. This is not necessary, but it serves to clarify the analysis somewhat. Note well that in this specification, no *a priori* mention is made of the order of the components of the surface; in fact, it is impossible to identify any of the three waves as being of second order (like the earlier ξ_{11}, say). Herein lies the differences in the method of ordering. Accordingly, the component a_3 will not be called a resonant secondary wave; a_1, a_2, and a_3 will be considered to be primary waves, since they all engage in growth to first order.

Through this analysis, it is shown that all three amplitudes vary periodically with time, appearing as a slow modulation of the three waves present, the modulation envelopes being given in terms of the Jacobian elliptic functions. There is no restriction on the time for which these results are valid as there is for the secular results of the perturbation solutions. In addition, it is shown that the total energy per unit surface area remains constant throughout the entire interaction, to second order. Since the wavelengths involved in these interactions are small compared to the wavelengths of pure gravity waves, the preceding results surely must be modified by the inevitable attenuation due to the influence of molecular viscosity in a real water-wave system, and suitable modifications to the analysis are made in § 5, by considering the action of viscosity to be a small perturbation on the assumed irrotational motion. Even with these modifications, a fairly large range of wave-numbers is found for which the effects of the interactions are profoundly significant.

While the results presented here are the first to show clearly the *continuing* energy transfer among the various resonant modes, this is not the only important reason for conducting this investigation. It may be expected that if the interactions occur rapidly enough, they could be of considerable importance in helping to explain some physically observable phenomena. The characteristic growth time for these second-order resonances is proportional to the (-1)-power of the geometric mean of the wave slopes, which is much quicker than the aforementioned tertiary characteristic times.

In the concluding section, it is shown that these interactions are of considerable importance in the problem of the initial development of the sea by a turbulent wind, and are equally important when considering the large wave-number details of the frequency and wave-number spectra of a random sea. In fact, it is possible to predict some kind of equilibrium range, or saturation range, for capillary-gravity and capillary waves in a random sea, and though detailed investigations of these questions are not yet made, indications of the significance of the interaction mechanisms are presented. It is suggested that further investigations of random wave phenomena at large wave-numbers will be incomplete without some accounting for these significant interactions.

Another reason, while not the primary one, but of no less importance, for investigating these interactions is to attempt partially to resolve some recent controversies on the resonance phenomenon. The perturbation analysis of Phillips has been severely criticized by Tick (1961) and Pierson (1961). Their chief criticism involves the interpretation of the linearly growing resonant tertiary term, which they quite properly identify in the language of non-linear mechanics as a secular term. They claim that this growth cannot occur for a long time because it would require a supply of energy not previously accounted for, whence they suggest that this wave can never become larger than third order. They further suggest two methods for the removal of this secular trend. The first (by Tick) is the use of the Krylov–Bogoliubov methods which were designed to remove secularity by further perturbation of the frequencies or relative phases. The second method, suggested by Pierson, is to include in the analysis terms resulting from higher orders in the perturbation scheme, the results being more or less equivalent to those obtained by the frequency perturbation procedure, at least for small times. The object of both these methods is the same, that is, to identify this secular term as the first term in a series expansion of an odd periodic function (Pierson 1961, p. 188). At the second order, it is found by the *non-perturbation* analysis presented in this paper that this secular term is indeed the leading term in the series expansion for the Jacobian elliptic function sn, which is the amplitude modulation function for the resonant wave. Furthermore, this result is found without recourse to higher-order terms, and since the analysis is of second order, it is not necessary to perturb the frequencies of the waves. The amplitudes of all three waves involved are of the same order, so here two waves can create a resonant wave which grows to the same order. (This maximum amplitude could not have been predicted had this analysis been carried out as a second-order perturbation analysis.) The energy required for this growth to first order does come, of course, from the two initial waves; this is shown in detail in a later section.

A further reason for this investigation is to shed some light on a curious singularity that arises in a steady-state perturbation analysis of finite amplitude one-dimensional gravity-capillary waves performed by Wilton (1915) and by Pierson & Fife (1961). They find a singularity at certain wave-numbers which results in infinite amplitudes of these critical modes. A re-interpretation in terms of the present analysis shows that these modes are self-resonant, the fundamental mode exchanging energy with its second harmonic in a periodic manner, both modes remaining finite, and so helps to resolve a long-standing difficulty.

This analysis suggests the further course for the third-order problem. The methods developed here can be applied to the third-order gravity wave problem, with suitable modifications necessary to account for the second-order frequency changes that appear from the inclusion of the third-order terms in the boundary conditions. A start was made in this direction by Benney (1962), but he stopped the analysis before considering the energies of the four interacting modes and could not predict the suspected periodicity of the time varying amplitudes. The algebra will be fearsome, the results rewarding.

2. The resonance conditions

Before investigating the dynamics of the interactions, it must first be determined whether or not there can be resonances at the second order. If two waves with wave-numbers $\mathbf{k}_1$ and $\mathbf{k}_2$ are present at some time t_0, the non-linear interaction between them will produce a wave with wave-number $\mathbf{k}_3$ such that $\mathbf{k}_3 = \mathbf{k}_1 \pm \mathbf{k}_2$. This wave will have a frequency equal to the sum or difference frequency $\sigma_1 \pm \sigma_2$, where the $\pm$ signs occur together. If this sum or difference frequency is equal to σ_3, the natural frequency of the $\mathbf{k}_3$ wave, then the $\mathbf{k}_3$ wave is excited at its natural frequency, and resonance can occur. If we can find triads satisfying this condition, then resonance is possible at the second order; if not, the analysis must be extended to third order, seeking resonances among quadruplets, as done by Phillips (1960) and Longuet-Higgins (1962) in the problem of purely gravity waves.

We therefore seek solutions to the following set of equations, hereafter called the resonance conditions

$$\left.\begin{aligned} \mathbf{k}_1 \pm \mathbf{k}_2 &= \pm\, \mathbf{k}_3, \\ \sigma_1 \pm \sigma_2 &= \pm\, \sigma_3, \\ \sigma_i^2 &= g k_i + T k_i^3, \end{aligned}\right\} \quad (i = 1, 2, 3) \tag{2.1}$$

with $T = T'/\rho$, T' being the coefficient of surface tension. The third equation of (2.1) with $i = 3$ insures that the $\mathbf{k}_3$ wave will indeed propagate as a free wave.

Equations (2.1) are suitably normalized by setting $\mathbf{K}_i = \mathbf{k}_i/k_m$, $n_i = \sigma_i/\sigma_m$, where $k_m = (g/T)^{\frac{1}{2}}$ and $\sigma_m = (4g^3/T)^{\frac{1}{4}}$ are the wave-number and frequency corresponding to the wave propagating at the minimum phase speed $c_m = (4gT)^{\frac{1}{4}}$ ($\approx 23{\cdot}2$ cm/sec for water). The corresponding wavelength and frequency are $\approx 1{\cdot}7$ cm and $13{\cdot}5$ cycles per second. Then (2.1) becomes

$$\left.\begin{aligned} \mathbf{K}_1 \pm \mathbf{K}_2 &= \pm \mathbf{K}_3, \\ n_1 \pm n_2 &= \pm n_3, \\ n_i^2 &= \tfrac{1}{2}(K_i + K_i^3). \end{aligned}\right\} \quad (i = 1, 2, 3). \tag{2.2}$$

Eliminating K_3 and the frequencies n_i and setting $\cos\theta = \mathbf{K}_1 . \mathbf{K}_2 / K_1 K_2$, where θ is the angle of intersection of the initial waves, (2.2) becomes

$$\begin{aligned} 8K_1^2K_2^2\cos^3\theta &+ 4(3K_1^3K_2 + 2K_1K_2 + 3K_1K_2^3)\cos^2\theta \\ &+ 2(1 + 4K_1^2 + 4K_2^2 + 3K_1^4 + 3K_2^4 + 6K_1^2K_2^2)\cos\theta \\ &- 6 + 4K_1K_2 - 6K_1^2K_2^2 + 3K_1^3K_2 + 3K_1K_2^3 - 6K_1^2 - 6K_2^2 \\ &- 4\{(K_1 + K_1^3)(K_2 + K_2^3)\}^{\frac{1}{2}}\left\{\frac{K_1 + K_2}{K_1K_2} + \frac{K_1^3 + K_2^3}{K_1K_2}\right\} = 0. \end{aligned} \tag{2.3}$$

For simplicity of calculation, plus signs are chosen wherever the $\pm$ sign occurs in (2.2), and all the results of this paper will concern 'sum-type' interactions. There is no loss of generality in doing so, however, because 'difference-type' interactions arise by interchanging the roles of a pair of the waves.

Now, for fixed initial wave-numbers, (2.3) is a cubic in ($\cos\theta$), and has at least one real root.† However, the only configurations physically realizable are those for which $|\cos\theta| \leqslant 1$. Suppose now that we fix $K_1 = 1$, say. Then for a sequence of values of K_2, the roots of (2.3) may be computed, yielding the angle of intersection $\theta = \theta(K_2; K_1)$. The results of these computations are shown in figure 1 for several fixed values of K_1 between $\frac{1}{3}$ and 3. This is a polar co-ordinate plot of the resonance angles θ and wave-numbers K_2 for nine different fixed values of K_1. The heavy lines labelled $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ demonstrate how this diagram is to be used. Here $\mathbf{K}_1 = (1, 0)$, and $\mathbf{K}_2$ was drawn along the line $\theta = 60°$ terminating on the curve labelled $K_1 = 1$. The line $\mathbf{K}_3$ is the closure of the triangle satisfying the resonance condition, indicating the magnitudes and directions of the wave-numbers in the resonant triad. Figure 2 is the result of a similar type of calculation for the special symmetric case where the two initial wave-numbers are equal in magnitude, showing the angle of intersection θ required for resonance. Note that in all of the cases shown in figures 1 and 2, there is a minimum wave-

† By applying Descartes' rule of signs, it can be shown that (2.3) has only one real positive root.

number K_2 for which resonance is possible. This is consistent with the earlier result that resonances cannot occur for the smaller wave-number gravity waves at this order, which was proved by Phillips (1960).

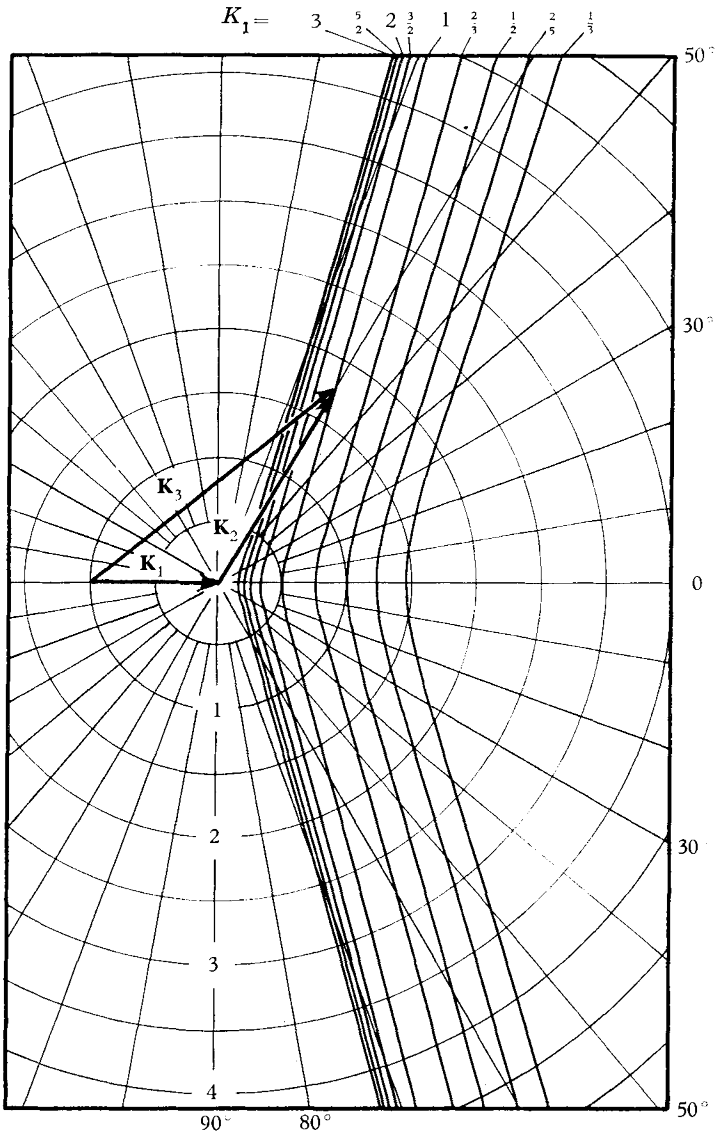

FIGURE 1. Solutions of the resonance condition.

For wave-numbers appreciably larger than k_m, gravity can be neglected in the resonance conditions (2.1), yielding the following simpler resonance conditions for pure capillary waves:

$$\mathbf{k}_1 \pm \mathbf{k}_2 = \pm \mathbf{k}_3, \quad \sigma_1 \pm \sigma_2 = \pm \sigma_3, \tag{2.4}$$

where $$\sigma_i^2 = Tk_i^3 \quad (i = 1, 2, 3).$$

This reduces to the following cubic equation for the capillary resonance angle θ

$$8\cos^3\theta + 12\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\cos^2\theta + 6\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)^2\cos\theta$$
$$= 6 - 3\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + 4\left\{\left(\frac{k_1}{k_2}\right)^{\frac{3}{2}} + \left(\frac{k_2}{k_1}\right)^{\frac{3}{2}}\right\}. \quad (2.5)$$

Again only sum-type interactions need be considered. This is considerably simpler than (2.3) because the initial wave-numbers k_1 and k_2 always appear in a ratio, say $k_2/k_1 = \alpha$. Figure 3 shows the solutions of (2.5) for α varying from

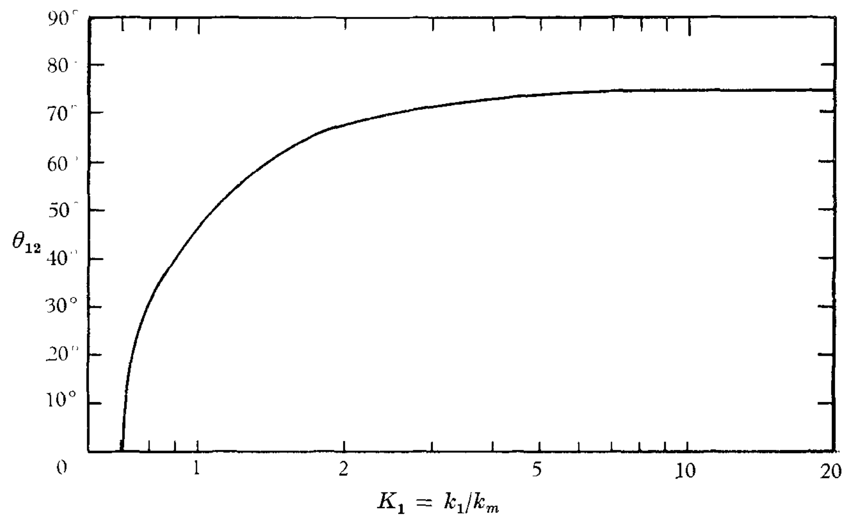

FIGURE 2. The resonance angle $\theta_{12} = 2\beta$ for the equal wave-number case, $k_1 = k_2$.

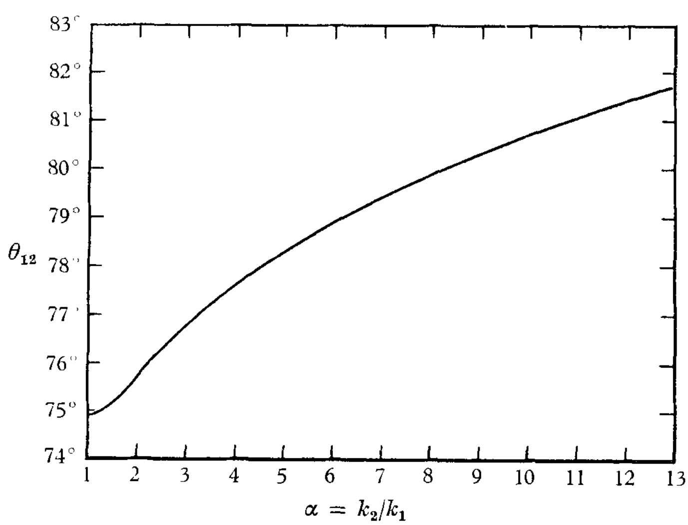

FIGURE 3. Solutions of the pure capillary resonance condition for various wave-number ratios $\alpha = k_2/k_1$.

1·0 to about 12. Note that for $\alpha = 1$, the equal wave-number case, the solution is $\cos\theta = 2^{\frac{1}{3}} - 1$ or $\theta = 74{\cdot}93°$, which is consistent with figure 2 for large K. It is remarkable that all these angles θ are around 75–80°.

3. The dynamical equations for resonant triads

Now that it has been established that resonance can indeed occur among triads of waves, we proceed to investigate the dynamics of the interactions. Choose a co-ordinate system such that the (x, y)-plane corresponds to the undisturbed free surface and z is increasing vertically. Then $z = \xi(\mathbf{x}, t)$ is the equation of the free surface, where $\mathbf{x} = (x, y)$. Suppose that three waves forming a resonant triad are present; then we write the equation of the free surface as

$$\xi(\mathbf{x}, t) = a_1(t)\cos\psi_1 + a_2(t)\cos\psi_2 + a_3(t)\cos\psi_3 \tag{3.1}$$

with $\psi_i = \mathbf{k}_i.\mathbf{x} - \sigma_i t + \epsilon_i$. The amplitudes $a_i(t)$ will be considered to be slowly varying functions of time (which will be specified formally later), and the wave-numbers have been chosen such that $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. The ϵ_i are phase angles, and will be constants in this analysis.

Since the analysis is only carried to second order, and we are neglecting the effects of molecular viscosity, we assume irrotational motion (see Phillips 1961) and therefore the existence of a velocity potential $\phi(\mathbf{x}, z, t)$ satisfying

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\phi = 0 \tag{3.2}$$

with $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The velocity is given by

$$\mathbf{u}(\mathbf{x}, z, t) = (\nabla\phi, \partial\phi/\partial z), \tag{3.3}$$

where ∇ is the two-dimensional operator

$$\nabla = (\partial/\partial x, \partial/\partial y).$$

When considering problems of surface waves, it is usual to suppose the arbitrary function of time in Bernoulli's equation to be merged in the value of $\partial\phi/\partial t$.† We assume the pressure above the free surface to be zero, so the pressure term in the Bernoulli equation will be that due to surface tension. Then the free surface dynamic boundary condition is

$$\frac{p}{\rho} + \frac{\partial\phi}{\partial t} + \tfrac{1}{2}u^2 + g\xi = 0, \tag{3.4}$$

evaluated at $z = \xi$. Continuity of pressure across the free surface requires that

$$\left(\frac{p}{\rho}\right)_\xi = -\frac{T'}{\rho}\left[\frac{\xi_{xx}(1+\xi_y^2) + \xi_{yy}(1+\xi_x^2) - 2\xi_{xy}\xi_x\xi_y}{(1+\xi_x^2+\xi_y^2)^{\frac{3}{2}}}\right],$$

which to second order is $(p/\rho)_\xi = -T\nabla^2\xi$.

† Lamb (1932, §227).

Then (3.4) may be rewritten as

$$(\partial\phi/\partial t)_\xi + \tfrac{1}{2}u_\xi^2 + g\xi - T\nabla^2\xi = 0. \tag{3.5}$$

The kinematic boundary condition is $D\xi/Dt = w_\xi$, or

$$\frac{\partial\xi}{\partial t} + (\nabla\phi)_\xi . (\nabla\xi) - \left(\frac{\partial\phi}{\partial z}\right)_\xi = 0. \tag{3.6}$$

Now, taking D/Dt of (3.5) and subtracting g times (3.6) gives the following combined boundary condition

$$\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right)_\xi + \left(\frac{\partial u^2}{\partial t}\right)_\xi + \mathbf{u}_\xi . \nabla(\tfrac{1}{2}u^2)_\xi + \tfrac{1}{2}w_\xi\left(\frac{\partial u^2}{\partial z}\right)_\xi - T\left[\left(\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right)\nabla^2\xi\right]_\xi = 0. \tag{3.7}$$

This is the same as Longuet-Higgins's equation (2.4) (1962) with the addition of the surface tension term. Also, since the water is assumed to be deep, we require

$$\mathbf{u}(\mathbf{x}, z, t) = \left(\nabla\phi, \frac{\partial\phi}{\partial z}\right) \to 0 \quad \text{as} \quad z \to -\infty. \tag{3.8}$$

A solution for ϕ satisfying (3.2), (3.5) and (3.8) will have the form

$$\phi(\mathbf{x}, z, t) = \frac{\sigma_1}{k_1} a_1(t)\, e^{k_1 z} \sin\psi_1 + \frac{\sigma_2}{k_2} a_2(t)\, e^{k_2 z} \sin\psi_2 + \frac{\sigma_3}{k_3} a_3(t)\, e^{k_3 z} \sin\psi_3 + O(ak)^2 \tag{3.9}$$

to be consistent with the classical infinitesimal slope results, where the terms of order $(ak)^2$ are precisely those which will be neglected in the ensuing analysis.

The combined boundary condition (3.7) may now be expanded in a Taylor series about the undisturbed free surface $z = 0$ in the following manner. Terms like $(\partial^2\phi/\partial t^2)_\xi$ are expanded as

$$\left(\frac{\partial^2\phi}{\partial t^2}\right)_\xi = \left(\frac{\partial^2\phi}{\partial t^2}\right)_0 + \xi\left[\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial t^2}\right)\right]_0 + \frac{\xi^2}{2}\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi}{\partial t^2}\right)\right]_0 + \ldots.$$

This expansion is applied to all the terms in (3.7), and we keep only those terms linear and quadratic in ϕ and ξ. Regrouping these terms with the linear terms on the left-hand side, we have the following combined boundary condition correct to second order

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} - T\frac{\partial}{\partial t}\nabla^2\xi = -\xi\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial t}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + T(\nabla\phi).\nabla(\nabla^2\xi), \tag{3.10}$$

where the terms are now to be evaluated at $z = 0$. Using (3.1) and (3.9) in (3.10), we arrive at a rather long and complicated expression involving the amplitudes $a_i(t)$ and the various circular function arguments ψ_i in different combinations. This expression appears in full in the Appendix (equation A 1).

The kinematic boundary condition (3.6) may also be expanded in a Taylor series about $z = 0$ in the same manner as (3.7), giving

$$\frac{\partial \xi}{\partial t} - \frac{\partial \phi}{\partial z} = \xi \frac{\partial^2 \phi}{\partial z^2} - (\nabla \phi).(\nabla \xi) \tag{3.11}$$

evaluated at $z = 0$. Using (3.1) and (3.9) in (3.11), we arrive at another expression involving the amplitudes a_i, which also appears in full in the appendix (equation A 2).

Equations (A 1) and (A 2) may be greatly simplified by using the manoeuvres which will now be outlined. Consider a term like $\sin \psi_1 \cos \psi_2$ which arises from the right-hand sides of (3.10) and (3.11). Using simple trigonometric identities, this can be written as

$$\sin \psi_1 \cos \psi_2 = \tfrac{1}{2} \sin (\psi_1 + \psi_2) + \tfrac{1}{2} \sin (\psi_1 - \psi_2),$$

a combination of sum and difference terms which arise quite naturally from the non-linear terms in the boundary conditions. Now

$$\psi_1 + \psi_2 = (\mathbf{k}_1 + \mathbf{k}_2).\mathbf{x} - (\sigma_1 + \sigma_2)t + \epsilon_1 + \epsilon_2.$$

Using the resonance conditions for sums,

$$\begin{aligned} \psi_1 + \psi_2 &= \mathbf{k}_3.\mathbf{x} - \sigma_3 t + \epsilon_1 + \epsilon_2 \\ &= \psi_3^{(+)} + \gamma_3. \end{aligned}$$

Thus the phase angle $\gamma_3 = \epsilon_1 + \epsilon_2 - \epsilon_3$. Also, we have for differences

$$\begin{aligned} \psi_1 - \psi_2 &= (\mathbf{k}_1 - \mathbf{k}_2).\mathbf{x} - (\sigma_1 - \sigma_2)t + \epsilon_1 - \epsilon_2 \\ &= \mathbf{k}_3.\mathbf{x} - \sigma_3 t + \epsilon_1 - \epsilon_2 \\ &= \psi_3^{(-)} + \delta_3. \end{aligned}$$

Here, the phase angle $\delta_3 = \epsilon_1 - \epsilon_2 - \epsilon_3$. Considering terms like $\sin \psi_1 \cos \psi_3$, we get similarly

$$\sin \psi_1 \cos \psi_3 = -\tfrac{1}{2} \sin (\psi_2^{(+)} + \gamma_2) + \tfrac{1}{2} \sin (\psi_2^{(-)} + \delta_2).$$

The first term on the right-hand side arises from the sum resonances, the second from differences. Thus each of the products arising from the non-linear terms for which the arguments of the circular functions are different can be interpreted as two terms, one of which is synchronization with the remaining wave of the triad for sum resonances. The other represents a difference-type interaction.

Now consider terms like $\sin \psi_1 \cos \psi_1$. This can be rewritten as $\frac{1}{2} \sin 2\psi_1$, a second harmonic. This propagates at a phase speed equal to that of the k_1 wave, and can be interpreted as a second term in a Stokes type expansion for a single finite amplitude wave.† The amplitudes of these 'bound secondaries' are always proportional to the amplitudes of their respective primaries, the proportionality being of order of wave slope, and their presence does not affect the phase speed at this order. Since these bound secondaries are small and they do not participate directly in the resonance, their effect will be neglected in the subsequent analysis.

† See, for instance, Lamb (1932, §250).

Along the same lines, when considering sum and difference interactions, we can see immediately that if three waves satisfy the sum resonance condition $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3$, the same physical configuration cannot satisfy the difference resonance conditions simultaneously. Since we are considering only sum type interactions in this paper, we can interpret the difference-type terms like $\sin(\psi_3^{(-)}+\delta_3)$ as bound secondary products of interaction, their amplitudes being of order of the geometric mean of the slopes times the geometric mean of the amplitudes of the first-order components from which they arise. Since these waves also do not participate directly in the resonance, their effects will be neglected for the same reasons as the bound secondaries.

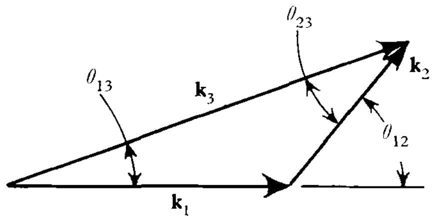

FIGURE 4. The geometry of the resonant triad configuration.

In the specification of the form of the free surface, the amplitudes $a_i(t)$ were considered to be slowly varying functions of time, so a further simplification arises by neglecting second time derivatives of the amplitudes. This requires that $\ddot{a}_i \ll \sigma_i^2 a_i$ which will be quantitatively justified later. After application of the above techniques to equations (A 1) and (A 2), the only circular functions remaining are those whose arguments are ψ_1, ψ_2 and ψ_3, with phase differences γ_1, γ_2 and γ_3, respectively. It then remains to sort out the coefficients of like harmonics, after which by some rather tedious algebra (A 1) and (A 2) may be combined to give the following three equations relating the coefficients of the three harmonics:

$$\left.\begin{aligned}\dot{a}_3\cos\psi_3 &= a_1a_2B_{12}\sin(\psi_3^{(+)}+\gamma_3),\\ \dot{a}_1\cos\psi_1 &= a_2a_3B_{23}\sin(\psi_1^{(+)}+\gamma_1),\\ \dot{a}_2\cos\psi_2 &= a_1a_3B_{13}\sin(\psi_2^{(+)}+\gamma_2).\end{aligned}\right\} \quad (3.12)$$

The constants B depend on the geometry of the triad, and are given by

$$\begin{aligned}B_{12} &= \frac{\sigma_1\sigma_2}{4\sigma_3^2}k_3\left[\frac{k_1}{k_3}\frac{\sigma_3^2}{\sigma_2}+\frac{k_2}{k_3}\frac{\sigma_3^2}{\sigma_1}-\frac{\sigma_1^2}{\sigma_2}-\frac{\sigma_2^2}{\sigma_1}\right.\\ &\qquad\left.-4\sigma_3\sin^2\tfrac{1}{2}\theta_{12}-T\cos\theta_{12}\left(\frac{k_1^3}{\sigma_1}+\frac{k_2^3}{\sigma_2}-\frac{k_3^2k_1}{\sigma_1}-\frac{k_3^2k_2}{\sigma_2}\right)\right],\\ B_{13} &= \frac{\sigma_1\sigma_3}{4\sigma_2^2}k_2\left[\frac{\sigma_1^2}{\sigma_3}-\frac{\sigma_3^2}{\sigma_1}+\frac{k_3\sigma_2^2}{k_2\sigma_1}-\frac{k_1\sigma_2^2}{k_2\sigma_3}\right.\\ &\qquad\left.+4\sigma_2\cos^2\tfrac{1}{2}\theta_{13}-T\cos\theta_{13}\left(\frac{k_3^3}{\sigma_3}-\frac{k_1^3}{\sigma_1}-\frac{k_2^2k_3}{\sigma_3}+\frac{k_2^2k_1}{\sigma_1}\right)\right],\\ B_{23} &= \frac{\sigma_2\sigma_3}{4\sigma_1^2}k_1\left[\frac{\sigma_2^2}{\sigma_3}-\frac{\sigma_3^2}{\sigma_2}+\frac{k_3\sigma_1^2}{k_1\sigma_2}-\frac{k_2\sigma_1^2}{k_1\sigma_3}\right.\\ &\qquad\left.+4\sigma_3\cos^2\tfrac{1}{2}\theta_{23}-T\cos\theta_{23}\left(\frac{k_3^3}{\sigma_3}-\frac{k_2^3}{\sigma_2}-\frac{k_1^2k_3}{\sigma_3}+\frac{k_1^2k_2}{\sigma_2}\right)\right].\end{aligned} \quad (3.13)$$

(The angles θ_{12}, θ_{13} and θ_{23} are shown in figure 4.)

For the coefficients of the circular functions to be non-trivial, we require that $\gamma_3 = \pi/2$. But $\gamma_3 = \epsilon_1+\epsilon_2-\epsilon_3$, and $\gamma_1 = -\epsilon_1-\epsilon_2+\epsilon_3$, $\gamma_2 = -\epsilon_1-\epsilon_2+\epsilon_3$, so then $\gamma_1 = \gamma_2 = -\frac{1}{2}\pi$. Then (3.12) becomes simply

$$\dot{a}_3 = a_1 a_2 B_{12}, \quad \dot{a}_1 = -a_2 a_3 B_{23}, \quad \dot{a}_2 = -a_1 a_3 B_{13}. \tag{3.14}$$

Before integrating this system of equations, a few comments are appropriate. If the analysis had been carried out *ab initio* as a perturbation analysis in the manner of Longuet-Higgins (1962), by writing ϕ, ξ, and $\mathbf{u}$ as power series, then the first of the equations (3.14) would arise as the coefficients of one of the quadratic terms.† In his analysis, the amplitudes of the primaries, say a_1 and a_2 are considered constants, and direct integration yields $a_3 = a_1 a_2 B_{12} t$. The obvious objections to this are that the results of the interaction yield a resonant wave which never stops growing, placing severe restriction on the time for which the analysis is valid. This is more severe at the second order here than it is in the previous third-order analyses since the growth is quicker at second order. Furthermore, there is no way to predict the maximum amplitude to which this resonant wave will grow. There can be no quarrel with the perturbation method as an initial value problem, and the initial rates of growth are certainly correct. The point of the argument is that when singularities such as these arise through perturbation methods, some delicacy is required in interpreting the results physically.

Finally, note the appearance of a_3 on the right-hand side of the last two equations of (3.14). Initially, or at time t_0, this amplitude is zero, whence $\dot{a}_1$ and $\dot{a}_2$ are both zero. These last two equations can never appear in a (second order) perturbation analysis, which requires $\dot{a}_1$ and $\dot{a}_2$ to be zero for ever. It is the purpose of the next section to investigate in detail the effects of these two equations.

4. The inviscid solution

Integration of the system (3.14) is straightforward. Let us suppose that $a_1(t_0) = \hat{a}_1$, $a_2(t_0) = \hat{a}_2$, $a_3(t_0) = 0$. Now, multiplying the first of (3.14) by a_3/B_{12}, the second by a_1/B_{23} and adding, we have

$$\frac{d}{dt}\left(\frac{a_1^2}{B_{23}}+\frac{a_3^2}{B_{12}}\right) = 0,$$

which upon integration gives

$$a_1^2 = \hat{a}_1^2\left(1-\frac{B_{23}}{B_{12}}\frac{a_3^2}{\hat{a}_1^2}\right). \tag{4.1}$$

Similarly, from the first and third of (3.14), we have

$$a_2^2 = \hat{a}_2^2\left(1-\frac{B_{13}}{B_{12}}\frac{a_3^2}{\hat{a}_2^2}\right). \tag{4.2}$$

† In fact, using Longuet-Higgins notation, if

$$\phi = \alpha\phi_{10}+\beta\phi_{01}+\alpha^2\phi_{20}+\alpha\beta\phi_{11}+\beta^2\phi_{02}+\dots,$$
$$\xi = \alpha\xi_{10}+\beta\xi_{01}+\alpha^2\xi_{20}+\alpha\beta\xi_{11}+\beta^2\xi_{02}+\dots,$$

then the coefficients of the $(\alpha\beta)$ terms yield precisely $\dot{a}_{11} = a_{10}a_{01}B_{12}$, where a_{10} and a_{01} are the amplitudes of ξ_{10} and ξ_{01}.

Returning these two results to the first of (3.14) we have

$$\left(\frac{da_3}{dt}\right)^2 = \hat{a}_1^2\hat{a}_2^2B_{12}^2\left[1-\frac{B_{13}}{B_{12}}\frac{a_3^2}{\hat{a}_2^2}\right]\left[1-\frac{B_{23}}{B_{12}}\frac{a_3^2}{\hat{a}_1^2}\right]. \tag{4.3}$$

This integrates directly in terms of the Jacobian elliptic functions with real parameter (the notation of Milne-Thomson 1950 will be used throughout), the result being

$$a_3(t) = \hat{a}_2(B_{12}/B_{13})^{\frac{1}{2}}\,\mathrm{sn}\,(\Xi|m), \tag{4.4a}$$

where

$$\Xi = \hat{a}_1(B_{12}B_{13})^{\frac{1}{2}}\,(t-t_0), \tag{4.4b}$$

and the parameter m is defined by

$$m = B_{23}\hat{a}_2^2/B_{13}\hat{a}_1^2. \tag{4.4c}$$

We shall assume that $m \leqslant 1$; if not, then use of Jacobi's real transformation† gives

$$a_3(t) = \hat{a}_1(B_{12}/B_{23})^{\frac{1}{2}}\,\mathrm{sn}\,[\hat{a}_2(B_{12}B_{23})^{\frac{1}{2}}\,(t-t_0)\,|\,m^{-1}],$$

which simply interchanges the roles of a_1 and a_2. Returning the solution (4.4) to (4.1) and (4.2), we have

$$\left.\begin{aligned} a_1(t) &= \hat{a}_1\,\mathrm{dn}\,(\Xi|m),\\ a_2(t) &= \hat{a}_2\,\mathrm{cn}\,(\Xi|m).\end{aligned}\right\} \tag{4.5}$$

Now, if $m = 1$, the Jacobian elliptic functions degenerate into hyperbolic functions, or

$$\left.\begin{aligned} a_1(t) &= \hat{a}_1\,\mathrm{sech}\,\Xi,\\ a_2(t) &= \hat{a}_2\,\mathrm{sech}\,\Xi,\\ a_3(t) &= \hat{a}_2\left(\frac{B_{12}}{B_{13}}\right)^{\frac{1}{2}}\tanh\Xi = \hat{a}_1\left(\frac{B_{12}}{B_{23}}\right)^{\frac{1}{2}}\tanh\Xi.\end{aligned}\right\} \tag{4.6}$$

This special case will be discussed later in more detail.

Equations (4.4) and (4.5) represent a system for which energy is shifted about periodically among the three waves present, there being no restriction on the time for which the solutions are valid. The following typical numerical example is illustrative of the general character of the solution. Suppose

$$k_1 = k_2 = k_m = 3{\cdot}667\,(\mathrm{cm})^{-1}.$$

The vaule of k_3 required for resonance is $k_3 = 6{\cdot}725\,(\mathrm{cm})^{-1}$, and the angle $\theta_{12} \doteq 47°$. Then $\theta_{13} = \theta_{23} = 23\frac{1}{2}°$, and $(B_{12}/B_{13})^{\frac{1}{2}} = 0{\cdot}958$, $(B_{12}B_{13})^{\frac{1}{2}} = 100{\cdot}3$. Now suppose $\hat{a}_1 = 0{\cdot}687$ mm, corresponding to a maximum slope of 0·252, and $\hat{a}_2 = 0{\cdot}487$ mm, corresponding to a maximum slope of 0·178. Then

$$\max|a_3| = \hat{a}_2(B_{12}/B_{13})^{\frac{1}{2}} = 0{\cdot}467\,\mathrm{mm},$$

which corresponds to a maximum slope of 0·314 (or a maximum steepness of 1/10, where steepness is $2a/\lambda$). Figure 5 shows the amplitudes plotted against the number of periods of the k_3 wave. The parameter $m = \frac{1}{2}$ for this case. Note in particular that the two initial waves create through this resonant interaction a wave whose slope becomes even greater than that of the initial waves, so the interaction certainly does create a wave whose slope grows above the second order.

† Milne-Thomson (1950), p. 19.

Before going on to the degenerate case (4.6), it is enlightening to consider the energy balance in this oscillatory system. The average energy per unit projected surface area can be written as the sum of the kinetic and potential energies and the energy due to the change of surface area in the following way:

$$E(t) = \lim_{S\to\infty}\left\{\frac{\rho}{2S}\int_S\int_{-\infty}^{\xi}\left[(\nabla\phi)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]dz\,ds + \frac{\rho g}{2S}\int_S \xi^2\,ds+\frac{T'}{S}\int_S[(1+\nabla\xi.\nabla\xi)^{\frac{1}{2}}-1]\,ds\right\}. \tag{4.7}$$

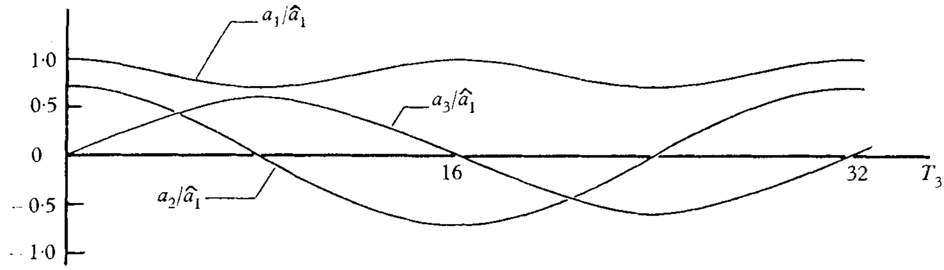

FIGURE 5. The time dependence of the amplitudes of the waves in the resonant triad. For this case, $k_1 = k_2 = k_m$, and $\hat{a}_1 = \sqrt{2}\,\hat{a}_2$. The interaction time T_I is approximately $8T_3$ for this case, T_3 being the period of the k_3-wave.

Using (3.1) and (3.9) it is easy to show that to second order (i.e. neglecting terms of order (slope)^2), (4.7) becomes

$$E(t) = \frac{\rho}{2}\left[\frac{\sigma_1^2 a_1^2}{k_1}+\frac{\sigma_2^2 a_2^2}{k_2}+\frac{\sigma_3^2 a_3^2}{k_3}\right]. \tag{4.8}$$

Now, returning to (3.14), multiplying the first equation by $\rho a_3\sigma_3^2/k_3$, the second by $\rho a_1\sigma_1^2/k_1$, and the third by $\rho a^2\sigma_2^2/k_2$ and adding, the result is

$$\frac{\rho}{2}\frac{d}{dt}\left[\frac{\sigma_1^2 a_1^2}{k_1}+\frac{\sigma_2^2 a_2^2}{k_2}+\frac{\sigma_3^2 a_3^2}{k_3}\right] = \rho a_1 a_2 a_3\left[\frac{\sigma_3^2 B_{12}}{k_3}-\frac{\sigma_1^2 B_{23}}{k_1}-\frac{\sigma_2^2 B_{13}}{k_2}\right], \tag{4.9}$$

or
$$\frac{dE(t)}{dt} = \rho a_1(t)\,a_2(t)\,a_3(t)\,\Delta. \tag{4.10}$$

Using the results (4.4) and (4.5) for the amplitudes, and writing

$$E(t_0) = \hat{E} = \tfrac{1}{2}\rho(\sigma_1^2\hat{a}_1^2/k_1+\sigma_2^2\hat{a}_2^2/k_2),$$

(4.10) can be integrated giving

$$E(t)/\hat{E} = 1+\Delta^*\,\mathrm{sn}^2(\Xi|m) \quad (m<1), \tag{4.11}$$

or
$$E(t)/\hat{E} = 1+\Delta^*\tanh^2\Xi \quad (m=1), \tag{4.12}$$

where
$$\Delta^* = \left(\frac{\sigma_3^2 B_{12}}{k_3}-\frac{\sigma_1^2 B_{23}}{k_1}-\frac{\sigma_2^2 B_{13}}{k_2}\right)\Bigg/\left(\frac{\sigma_1^2 B_{23}}{k_1}+\frac{\sigma_2^2 B_{13}}{k_2}\right)$$
$$= \Delta\Bigg/\left(\frac{\sigma_1^2 B_{23}}{k_1}+\frac{\sigma_2^2 B_{13}}{k_2}\right). \tag{4.13}$$

Then
$$|[E(t)-\hat{E}]/\hat{E}| \leqslant \Delta^*.$$

On purely physical grounds we expect that, since there is no energy input and the effects of viscosity are neglected so far, the energy of the system must remain constant, equal to $\hat{E}$. This requires that the constant Δ^* be identically zero for all resonance configurations. Algebraic proof of this identity is desirable, mainly to serve as a check on the algebraic coefficients B, but is a daunting undertaking due to the complexity of the coefficients in all but the simplest configurations. Therefore the number Δ^* was computed for all the configurations shown in figures 1 and 2. For the purposes of this computation, the cgs system of units was used, with $g = 980$ and $T = 72{\cdot}88$ (corresponding to water at 20 °C.) This gives $C_m = 23{\cdot}12\,\text{cm/sec}$, $k_m = 3{\cdot}667\,(\text{cm})^{-1}$, and $\sigma_m = 84{\cdot}78\,\text{rad/sec}$. In all of the hundreds of calculations, the small departures of Δ^* from zero can be ascribed to truncation errors in the machine. Furthermore, several special cases which lend themselves to particularly easy analytical calculation emphasize the fact that Δ^* is zero.

First, consider the case where $k_1 = k_2 = k_m/\sqrt{2}$. Then the resonance condition (2.3) becomes $\cos^3\theta_{12} + 5\cos^2\theta_{12} + 8\cos\theta_{12} - 14 = 0$. One exact root is $\cos\theta_{12} = 1$, the other two being the physically unrealistic complex conjugate pair $-3 \pm i\sqrt{5}$. Therefore, $\theta_{12} = \theta_{23} = \theta_{13} = 0$, and $k_3 = 2k_1$. Also $\sigma_1 = \sigma_2 = \sigma_3/2$, so from (3.13) after some algebra, we see that $B_{12} = B_{13} = B_{23} = \frac{1}{2}\sigma_1 k_1$. Then

$$\Delta^* = (\sigma_1^3 - \tfrac{1}{2}\sigma_1^3 - \tfrac{1}{2}\sigma_1^3)/\sigma_1^3 \equiv 0.$$

Secondly, we consider wave-numbers large enough for the pure capillary resonance condition (2.4) to hold, and investigate the case where $\alpha = 1$, i.e. $k_1 = k_2$. The angle required for resonance (see § 2) is $\theta_{12} = \cos^{-1}(2^{\frac{1}{3}} - 1) = 74{\cdot}93°$. Then simple calculation shows that $k_3 = 2^{\frac{2}{3}}k_1$ and $\sigma_1 = \sigma_2 = \sigma_3/2$. Then

$$B_{12} = (2^{-\frac{2}{3}} + 2^{-1} - 2^{-2} - 2^{-\frac{1}{3}})\,\sigma_1 k_1 = 0{\cdot}086\sigma_1 k_1,$$

and
$$B_{23} = B_{13} = (2^{-\frac{5}{3}} + 2^{\frac{2}{3}} - 2^{\frac{1}{3}} - 1)\,\sigma_1 k_1 = 0{\cdot}108\sigma_1 k_1.$$

So
$$\frac{B_{12}}{B_{23}} = 2^{-\frac{1}{3}}\left[\frac{1 + 2^{-\frac{5}{3}} - 2^{-\frac{1}{3}} - 2^{-\frac{2}{3}}}{1 - 2^{-\frac{5}{3}} + 2^{-\frac{1}{3}} - 2^{\frac{2}{3}}}\right] = 2^{-\frac{1}{3}}.$$

Then
$$\Delta^* = \left(\frac{\sigma_3^2 B_{12}}{k_3} - \frac{2\sigma_1^2 B_{23}}{k_1}\right)\Big/\left(2\,\frac{\sigma_1^2 B_{23}}{k_1}\right)$$
$$= 2^{\frac{1}{3}}\,\frac{B_{12}}{B_{23}} - 1 \equiv 0.$$

The two exact cases calculated above, together with the machine calculations, while not a general proof of the identity $\Delta^* \equiv 0$, lend strong support to its validity. So *the total energy per unit projected surface area in the resonant triad remains constant*, and equal to $\hat{E}$. Therefore, in the inviscid case, the resonant interactions are energy conserving; they simply redistribute energy among the three modes in a periodic manner.

Writing
$$E_1(t) = \frac{\rho\sigma_1^2 a_1^2(t)}{2k_1}, \text{ etc.,}$$

we can plot the time history of the energy of the various interacting modes. Figure 6 shows this for the same configuration as given in figure 5. The vertical height of the shaded section represents the total energy contained in the k_2 mode at any time. The portions above and below this section are the energies contained in the k_1 and k_3 modes, respectively. During the initial period of interaction, the energy in k_3 increases at the expense of both k_1 and k_2. Eventually, the energy in the k_2 mode disappears, at which time the growing mode has acquired its greatest energy. Then the direction of energy transfer is reversed;

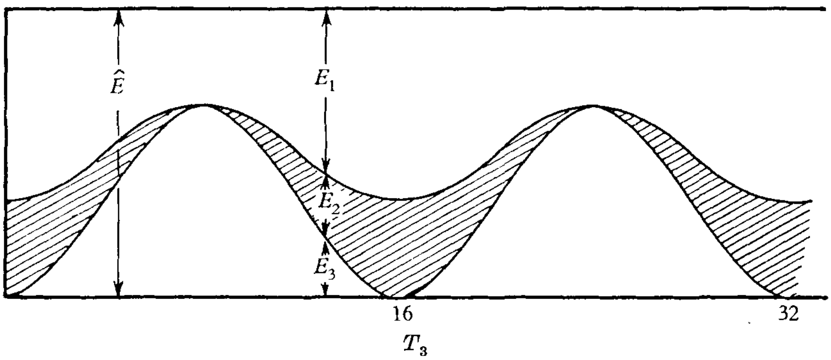

FIGURE 6. Time dependence of the energies of the various interacting modes for the same configuration as Figure 5.

the k_1 and k_2 modes increase at the expense of the k_3 mode until the initial configuration is reached again. The period of this energy transfer is

$$T_E = 2K(m)\,(\hat{a}_1^2 B_{12} B_{13})^{-\frac{1}{2}},$$

where $K(m)$ is the complete elliptical integral of the first kind. Note also that the energy in the k_1 mode never completely disappears, since dn $(\Xi|m) > 0$ for $m < 1$. For $m = 1$, dn u becomes sech u which $\to 0$ as $u \to \infty$. This is the degenerate case described by (4.6).

It seems prudent that before investigating this case, a physical interpretation of the parameter m be given in terms of the various rates of growth (or decay) in the problem. Consider the second and third of (3.14).

$$\dot{a}_1 = -a_2 a_3 B_{23}, \quad \dot{a}_2 = -a_1 a_3 B_{13}.$$

Simple algebra gives the result

$$\frac{d}{dt}\left(\frac{a_1}{\hat{a}_1}\right)^2 \Big/ \frac{d}{dt}\left(\frac{a_2}{\hat{a}_2}\right)^2 = B_{23}\hat{a}_2^2/B_{13}\hat{a}_1^2 = m. \tag{4.14}$$

Therefore, m is the ratio of the fractional rate of growth (or decay) of the square of the amplitude of the k_1 mode to that of the k_2 mode at any time. Or in other words, m is the ratio of the effectiveness of the two initial waves in sharing in the energy transfer mechanism; if $m < 1$, the k_2 mode can transfer energy into the k_3 mode at a greater proportional rate than the k_1 mode can, explaining why (in figure 6) the energy of the k_1 mode never disappears. This also justifies a previous statement that $m > 1$ simply reverses the roles of the k_1 and k_2 modes. Note that this is not an energy criterion, but rather an *energy transfer rate* criterion; the wave containing initially the greatest energy is not necessarily the one which never loses its energy completely.

Now for $m = 1$, both initial waves play identical parts in the energy transfer, losing energy at the same fractional rate. As a consequence, the energy transfer is no longer periodic, but the system approaches an asymptotic configuration in which the initial waves have disappeared, all of their energy having been fed into the monotonically growing k_3 wave. The total energy, of course, is still constant, the energy integral (4.12) applying in this case.

While it is not necessary for the two initial wavelengths to be equal in order to generate this non-periodic interaction for which $m = 1$, investigation of this configuration is analytically less complicated, and some results directly applicable to some observable physical phenomena can be obtained. If $k_1 = k_2$, then it is immediately obvious that $\sigma_1 = \sigma_2 = \sigma_3/2$, and $k_3 = 2k_1 \cos\beta$, where $\beta = \theta_{13} = \theta_{23}$ and is half the angle θ_{12} required for resonance in figure 2. Then from (4.13) with $\Delta^* = 0$, we see that

$$B_{13} = B_{23} = (\cos\beta)^{-1} B_{12}. \tag{4.15}$$

Then $m = \hat{a}_2^2/\hat{a}_1^2$, and the amplitudes are given by

$$\left.\begin{aligned} a_1(t) &= \hat{a}_1 \,\mathrm{dn}\,[\hat{a}_1 B_{12} (\cos\beta)^{-\frac{1}{2}} (t-t_0)|m], \\ a_2(t) &= \hat{a}_2 \,\mathrm{cn}\,[\hat{a}_1 B_{12} (\cos\beta)^{-\frac{1}{2}} (t-t_0)|m], \\ a_3(t) &= \hat{a}_2 (\cos\beta)^{\frac{1}{2}} \,\mathrm{sn}\,[\hat{a}_1 B_{12} (\cos\beta)^{-\frac{1}{2}} (t-t_0)|m], \end{aligned}\right\} \tag{4.16}$$

or for $\hat{a}_1 = \hat{a}_2$

$$\left.\begin{aligned} a_1(t) = a_2(t) &= \hat{a}_1 \,\mathrm{sech}\,[\hat{a}_1 B_{12} (\cos\beta)^{-\frac{1}{2}} (t-t_0)], \\ a_3(t) &= \hat{a}_1 (\cos\beta)^{\frac{1}{2}} \tanh\,[\hat{a}_1 B_{12} (\cos\beta)^{-\frac{1}{2}} (t-t_0)]. \end{aligned}\right\} \tag{4.17}$$

Figure 7 shows the values of the constants B. We can now calculate a typical characteristic interaction time T_I, being (for (4.17)) the time required for the growing wave to reach approximately 76 % of its maximum amplitude. At this time, the initial waves will have decayed to about 65 % of their initial amplitude. This happens when $T_I = 1/\hat{a}_1 B_{12} (\cos\beta)^{-\frac{1}{2}}$. In terms of the number of elapsed periods of the initial wave, this is

$$\begin{aligned} N_I = \frac{\sigma_1 T_I}{2\pi} &= \frac{\sigma_1 k_1 (\cos\beta)^{\frac{1}{2}}}{2\pi B_{12}} (\hat{a}_1 k_1)^{-1} \\ &= \gamma(k_1)(\hat{a}_1 k_1)^{-1}, \end{aligned} \tag{4.18}$$

so we see that the interaction time is inversely proportional to the maximum slope of the initial waves. Figure 8 shows N_I as a function of k_1 for initial waves having a maximum slope of about 0·3, indicating that for steep waves, the interactions are quite rapid at the second order. The slowly varying amplitude approximation made in § 3 should be examined. In order to neglect second derivatives of amplitudes, we require that $\ddot{a} \ll \sigma^2 a$. Using the solutions (4.17) this means that

$$\hat{a}_1^3 B_{12}^2 (\cos\beta)^{-1} \ll \hat{a}_1 \sigma_1^2,$$

or,

$$(\hat{a}_1 k_1)^2 B_{12}^2 / k_1^2 \sigma_1^2 \cos\beta \ll 1. \tag{4.19}$$

If we assume that the maximum slope is $\sim 0{\cdot}3$, then for wave-numbers near k_m, the inequality (4.19) is satisfied to about 2 orders of magnitude (10^{-2}), whereas

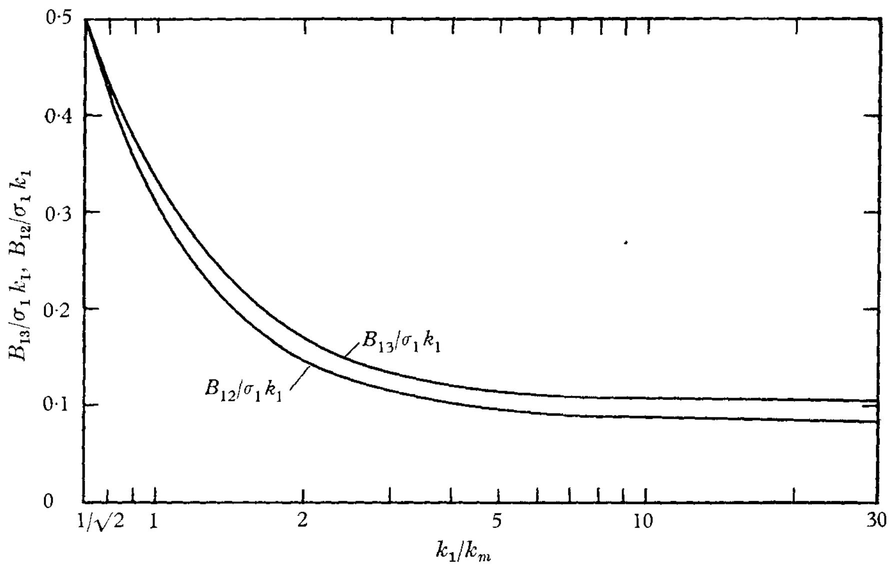

FIGURE 7. The constants B_{12} and B_{13} for the equal wave-number case. For large $k_1 = k_m/\sqrt{2}$, $B_{12} = B_{13}$. For large k_1, the pure capillary case, $B_{13}/B_{12} = 2^{\frac{1}{3}}$.

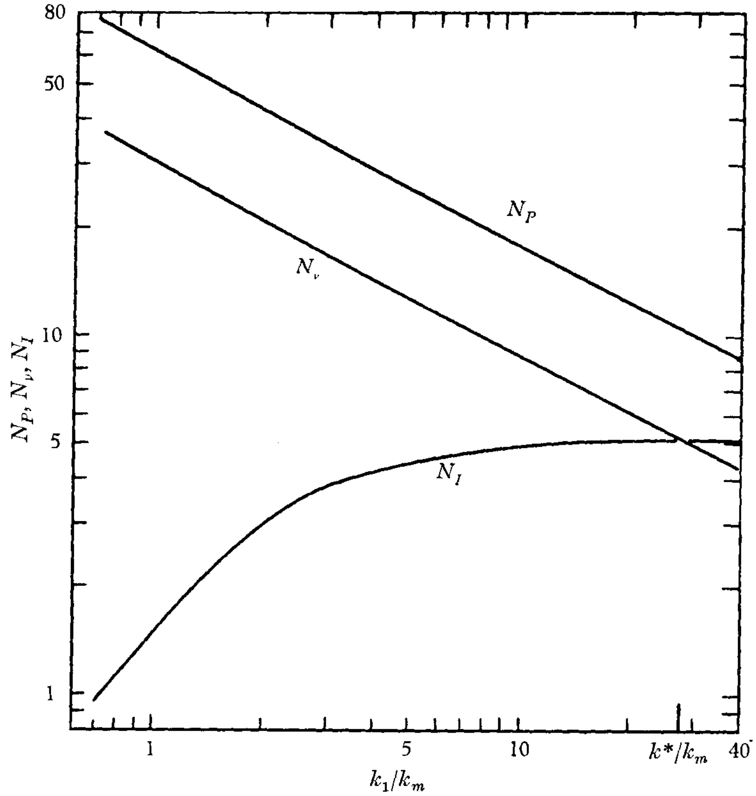

FIGURE 8. A comparison of the characteristic times involved in the interactions for the equal wave-number case. k^* is the wave-number for which the viscous effects are as important as the effects of the interaction. N_P, N_ν, N_I are measured in number of periods of the *initial* waves.

for the larger wave-number pure capillary case, the $\ll$ sign can be interpreted as about 3 orders of magnitude. The slowly varying approximation seems to be good enough for the purposes of this second-order analysis.

5. The influence of molecular viscosity

While the results of the preceding section certainly demonstrate the mechanism of these continuing periodic interactions, the application of these results to a real water-wave problem would by no means be complete without some estimate about the effects of molecular viscosity on the interaction. For the larger wave-lengths involved in these interactions, say around 2·5 cm, the action of viscosity to first order in the absence of resonant interactions is to attenuate the waves exponentially as they progress. The time constant $(2\nu k^2)^{-1}$ is about 75 periods for the larger waves near k_m and considerably smaller for the smaller capillary waves. This, however, is still considerably longer than the typical interaction times given previously, and for a moderate range of wave-numbers, we might consider the viscous effects as a small perturbation on the interaction.

In his (more complete) analysis of the viscous effects on finite amplitude waves, Longuet-Higgins (1953) found that the fundamental second-order effects are the presence of a small mass transport velocity in the regions near the physical boundaries, and the presence of a second-order mean vorticity field, affecting the mean motion to this order, but leaving the oscillatory properties unchanged. Since we are considering the viscous action to be a small perturbation in this particular estimate, the second-order effects found by Longuet-Higgins will be neglected.

This estimate can be made in the following way. In the absence of interactions, the amplitudes are given by

$$a_i(t) = a_i(t_0)\exp\left[-2\nu k_i^2(t-t_0)\right],$$

or

$$\dot{a}_i = -2\nu k_i^2 a_i.\dagger \tag{5.1}$$

Since the characteristic time for viscous attenuation is longer than the interaction times, we may speculate that the viscous effects will be a small temporal perturbation on the interacting system (3.14). By virtue of the form of (5.1), we may simply add these attenuation terms onto the interaction terms as a first approximation. Then (3.14) becomes

$$\left.\begin{aligned} \dot{a}_3 &= a_1 a_2 B_{12} - 2\nu k_3^2 a_3, \\ \dot{a}_1 &= -a_2 a_3 B_{23} - 2\nu k_1^2 a_1, \\ \dot{a}_2 &= -a_1 a_3 B_{13} - 2\nu k_2^2 a_2. \end{aligned}\right\} \tag{5.2}$$

Before attempting to integrate this system of equations, it is enlightening to consider the effects of the decay terms on the energy integral. Performing the same set of manipulations that led to (4.9), we get

$$\frac{d}{dt}\left(\frac{\sigma_1^2 a_1^2}{k_1} + \frac{\sigma_2^2 a_2^2}{k_2} + \frac{\sigma_3^2 a_3^2}{k_3}\right) = 2a_1 a_2 a_3 \Delta - 4\nu(k_1\sigma_1^2 a_1^2 + k_2\sigma_2^2 a_2^2 + k_3\sigma_3^2 a_3^2). \tag{5.3}$$

† See Lamb (1932, § 348).

But the inclusion of viscosity in no way alters the kinematic constant Δ (see (4.9) and (4.10)), so we use a result of part 4 and set $\Delta = 0$. The resulting equation is

$$\frac{d}{dt}\left(\frac{\sigma_1^2 a_1^2}{k_1}+\frac{\sigma_2^2 a_2^2}{k_2}+\frac{\sigma_3^2 a_3^2}{k_3}\right) = -4\nu(k_1\sigma_1^2 a_1^2 + k_2\sigma_2^2 a_2^2 + k_3\sigma_3^2 a_3^2). \tag{5.4}$$

But this equation is satisfied identically by (5.1), the viscous attenuation in the absence of interaction, and the left-hand side of (5.4) is equal to $(2/\rho)\,dE/dt$. Therefore, it may be concluded that to the order considered in this paper, the decay rate of the total energy per unit projected surface area is exactly the same as it would be if there were no interactions taking place at all. The difference must appear at the third order or higher.

The above energy considerations suggest a possible method for the integration of equations (5.2). Exact solution of this system is quite difficult and, as will be shown later, turns out to be unnecessary for our purposes here. Since we are considering the viscous effects to be small, their influence might be considered to be a perturbation on the frequency of modulation of the amplitudes, as is suggested by the non-linear asymptotic methods of Krylov–Bogoliubov–Mitropolski.† As a first approximation, we let

$$\left.\begin{aligned} a_1(t) &= \hat{a}_1\,\mathrm{dn}\,(\chi_1(t)|m)\exp\{-2\nu k_1^2(t-t_0)\},\\ a_2(t) &= \hat{a}_2\,\mathrm{cn}\,(\chi_2(t)|m)\exp\{-2\nu k_2^2(t-t_0)\},\\ a_3(t) &= \hat{a}_2(B_{12}/B_{13})^{\frac{1}{2}}\,\mathrm{sn}\,(\chi_3(t)|m)\exp\{-2\nu k_3^2(t-t_0)\},\end{aligned}\right\} \tag{5.5}$$

where χ_1, χ_2 and χ_3 are all given in the first approximation as $\chi_1 = \chi_2 = \chi_3 = \Xi$ (see (4.5)). Substitution of (5.5) into the first of (5.2) gives an equation for $\chi_3(t)$ of the form

$$d\chi_3/dt = \hat{a}_1(B_{12}B_{13})^{\frac{1}{2}}\exp\{2\nu(k_3^2-k_1^2-k_2^2)(t-t_0)\},$$

integration of which gives

$$\chi_3(t) = \hat{a}_1(B_{12}B_{13})^{\frac{1}{2}}(t-t_0)\left[\sum_{n=1}^{\infty}\frac{(\Gamma_3(t-t_0))^{n-1}}{n!}\right], \tag{5.6}$$

with $\Gamma_3 = 2\nu(k_3^2-k_1^2-k_2^2)$. So as a second approximation for some small time, we have

$$\chi_3(t) = \Xi[1+\nu(k_3^2-k_1^2-k_2^2)(t-t_0)+\ldots]. \tag{5.7a}$$

Similar manipulations for χ_1 and χ_2 give

$$\left.\begin{aligned}\chi_1(t) &= \Xi[1+\nu(k_1^2-k_2^2-k_3^2)(t-t_0)+\ldots],\\ \chi_2(t) &= \Xi[1+\nu(k_2^2-k_1^2-k_3^2)(t-t_0)+\ldots].\end{aligned}\right\} \tag{5.7b}$$

Therefore, the apparent effects of viscosity are twofold: first, to produce an exponential attenuation of the three interacting amplitudes of the same amount that would be expected in the absence of interactions, and secondly, to modify the periods of energy interchange among the interacting modes by some small amount which will be discussed below. The amplitudes of the three modes may now be written approximately as

$$\left.\begin{aligned} a_1(t) &= \hat{a}_1\,\mathrm{dn}\,[\Xi(1+\nu(k_1^2-k_2^2-k_3^2)(t-t_0)+\ldots)|m]\times\exp\{-2\nu k_1^2(t-t_0)\},\\ a_2(t) &= \hat{a}_2\,\mathrm{cn}\,[\Xi(1+\nu(k_2^2-k_1^2-k_3^2)(t-t_0)+\ldots)|m]\times\exp\{-2\nu k_2^2(t-t_0)\},\\ a_3(t) &= \hat{a}_2(B_{12}/B_{13})^{\frac{1}{2}}\,\mathrm{sn}\,[\Xi(1+\nu(k_3^2-k_1^2-k_2^2)(t-t_0)+\ldots)|m]\\ &\qquad\times\exp\{-2\nu k_3^2(t-t_0)\}.\end{aligned}\right\} \tag{5.8}$$

† See, for instance, Bogoliubov & Mitropolski (1961).

Therefore, the amplitudes are no longer periodic in time as they were in the inviscid case, but quasi-periodic, with their relative phases differing slowly in time. In order to discuss the time scales of these phase changes, we again turn to the symmetrical equal wave-number case ($k_1 = k_2$) for obvious analytical simplicity.

The addition of viscosity has introduced two kinds of time scales in addition to the interaction time N_I defined in the preceding section. The first kind is connected with the viscous attenuation of the amplitudes, and may be related to the time constant $(2\nu k^2)^{-1}$ appearing in the exponential terms. The smallest of these times is that associated with the largest wave-number in the configuration, k_3, which is the most rapidly attenuated wave. Then $T_\nu = (2\nu k_3^2)^{-1}$, or, in terms of the number of periods of the *initial* waves, $N_\nu = \sigma_1/4\pi\nu k_3^2$. For the symmetrical case, with $k_1 = k_2 = k_3/2\cos\beta$,

$$N_\nu = \frac{\sigma_1}{16\pi\nu k_1^2\cos^2\beta} = \frac{(gk_1+Tk_1^3)^{\frac{1}{2}}}{16\pi\nu k_1^2\cos^2\beta}. \tag{5.9}$$

The other kind of characteristic time, a phase shift time, arises as follows: for the symmetrical case, (5.8) may be rewritten

$$\left.\begin{aligned}
a_1(t) &= \hat{a}_1\,\mathrm{dn}\left[\frac{\hat{a}_1B_{12}}{(\cos\beta)^{\frac{1}{2}}}(t-t_0)\{1-2\nu k_1^2(1+\cos 2\beta)(t-t_0)\}\,|\,m\right]\\
&\qquad\qquad\times\exp\{-2\nu k_1^2(t-t_0)\},\\
a_2(t) &= \hat{a}_2\,\mathrm{cn}\left[\frac{\hat{a}_1B_{12}}{(\cos\beta)^{\frac{1}{2}}}(t-t_0)\{1-2\nu k_1^2(1+\cos 2\beta)(t-t_0)\}\,|\,m\right]\\
&\qquad\qquad\times\exp\{-2\nu k_1^2(t-t_0)\},\\
a_3(t) &= \hat{a}_2(\cos\beta)^{\frac{1}{2}}\,\mathrm{sn}\left[\frac{\hat{a}_1B_{12}}{(\cos\beta)^{\frac{1}{2}}}(t-t_0)\{1+2\nu k_1^2(\cos 2\beta)(t-t_0)\}\,|\,m\right]\\
&\qquad\qquad\times\exp\{-2\nu k_3^2(t-t_0)\}.
\end{aligned}\right\} \tag{5.10}$$

So a characteristic time for phase changes due to the action of viscosity is $T_P = \{2\nu k_1^2(1+\cos 2\beta)\}^{-1}$, being the smaller of the two 'time constants' in the quasi-periodic arguments of (5.10). This may also be written in terms of the number of periods of the *initial* waves as

$$N_P = \frac{\sigma_1}{4\pi\nu k_1^2(1+\cos 2\beta)} = \frac{(gk_1+Tk_1^3)^{\frac{1}{2}}}{8\pi\nu k_1^2\cos^2\beta}. \tag{5.11}$$

Comparing this with (5.9), we see that $N_P = 2N_\nu$. The characteristic times N_P and N_ν are shown in figure 8. Notice that there is an appreciable range of wave-numbers for which the interaction time is considerably smaller than the viscous decay time, so we might draw the following conclusion. For a moderate range of wave-numbers, say $k_m/\sqrt{2} < k < k^*$, the effects of the resonant interactions are considerably more important than the effects of viscosity. For the quasi-periodic a_3, the period of energy transfer is slowly decreasing with time, while the rate at which energy is shifted in and out of a_1 and a_2 is slowly increasing with time. But the time characteristic of this change is twice as long as that characteristic of the viscous attenuation of the smallest wavelength present in the triad. If we restrict ourselves to times of this order, and wave-numbers of this range, then the principal effect of viscosity is to attenuate the waves at the same rate that they would be attenuated in the absence of resonant interaction, and to gradually retard the rate of (or increase the period of) energy interchange. Furthermore, the resonant interactions have no effect on the rate of decay of the total energy to the order considered here. It is certainly true that for times larger than these and for wave-numbers greater than these, the perturbation type analysis presented here fails, and we must then take recourse to a full second-order analysis; however, for practical purposes, the analysis above is quite sufficient, and indicates the general nature of the phenomenon.

6. Discussion and concluding remarks

It may seem curious that so much attention has been given to the equal wave-length case in the preceding several sections. While the analysis is somewhat simpler for this case, an important justification for this closer scrutiny is that this case can arise quite naturally through previously studied physical phenomena. It is the purpose of this concluding section to interpret the results of the capillary-gravity interactions in the context of the problems of the creation and growth of ocean waves, and the redistribution of energy in the various non-linear random wave spectra.

The problem of the initial generation of waves by a turbulent wind passing over a relatively undisturbed water surface has been investigated in some detail by several authors, the modern approach being initiated by Phillips (1957), and continued by Miles in several subsequent papers. In his 1962 paper, Miles presents a review of the mechanisms proposed for the transfer of energy from the wind, among which he includes the wind-wave resonance mechanism of Phillips.

One of Miles's principal reasons for suspecting that turbulent pressure fluctuations acting alone are inadequate for the generation of short water waves is that the fluctuations at such small wavelengths are relatively weak and that they are convected downstream too rapidly to account for the *straight crested* waves that are sometimes observed. He then proceeds to investigate in some numerical detail the growth of short waves by a shear flow having a profile corresponding to the mean flow in a turbulent boundary layer, a mechanism originally proposed by Brooke Benjamin (1959). He suggests that the energy transfer from wind to waves through the action of the viscous Reynolds stress is of considerable importance. He presents graphically the typical growth rates as a function of wavelength, using the friction velocity of the shear flow as a parameter.

The results of Miles's calculations indicate that for wavelengths of the order of 1–2 cm, and for moderate wind speeds, typical growth times (time constants, say) are of the order of $\frac{1}{2}$ to 1 sec. In terms of the number of periods of the water waves, then, the growth times due to energy input from the wind are of order 10 periods or greater. But from figure 8 it can be seen that typical interaction times N_I for wavelengths of this order are in the neighbourhood of 2 to 3 wave periods of the initial waves, say, so that we may conclude that in this wave-length range, at least, the resonant interactions are a more efficient mechanism for the growth of certain wavelength waves than is the energy input from the wind. This implies that the shortest waves can acquire energy more rapidly by interaction than they can from the wind alone.

Returning to Miles's objections to the Phillips wind-wave resonance phenomenon, it should be noted that while the results of the present paper certainly cannot account for the dynamics of the energy interchange between the turbulent pressure fluctuations and the waves, they can account for the occasional appearance of straight-crested waves travelling with the wind. Phillips (1957) proposed that turbulent pressure fluctuations of eddy size k_1^{-1} (measured in the direction

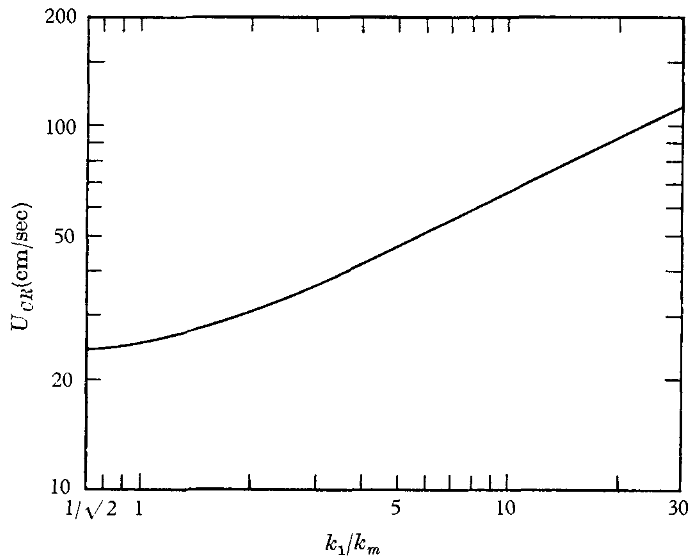

FIGURE 9. The critical convection speed U_{CR} for which the two resonance conditions are satisfied simultaneously.

of the wind) being convected downstream by the wind with a convection velocity U_c can result in a resonant growth of water waves of wave-number k_1 travelling in rhomboidal patterns oblique to the wind. In that case the angle α between the normals to the wave crests in the direction of propagation and the direction of the wind is given by the expression $|\alpha| = \sec^{-1}(U_c/C_1)$, C_1 being the phase velocity of the generated waves. Now, for certain values of the convection velocity U_c, it is possible for the wind-wave resonance angle α to be identical to the wave-wave resonance angle β defined in a previous section for the equal wave-number case. Denoting this critical convection velocity by U_{CR}, then we find that $U_{CR}(k_1) = C_1(k_1) \sec\beta(k_1)$, $\beta(k_1)$ being determined from figure 2 as a function of wave-number k_1 $(\beta = \frac{1}{2}\theta_{12})$, and C_1 being equal to $\sigma_1/k_1 = (g/k_1 + Tk_1)^{\frac{1}{2}}$. Figure 9 shows the results of this calculation, giving the values of U_{CR} required for simultaneous wind-wave resonance and wave-wave resonance.

For these particular configurations then, the following rough model suggests itself for the time sequence of events in the initial generation of waves by wind when the *two* resonance mechanisms are operating. Initially, the wind will raise two waves of wave-number k_1 at angles $\pm\beta$ to the wind. These waves, according to the Phillips resonance mechanism, will grow linearly for some small time.

Now, since the rates of growth for this mechanism depend critically on a detailed knowledge of the turbulent flow field of the wind above the water, information for which there is a remarkable (and quite unfortunate) scarcity, we can do no better at this time than to assume that the rates of growth are those calculated by Miles, just mentioned previously. Indeed this assumption cannot be too far off; the one-dimensional wave propagating analysis of Brooke Benjamin and Miles could be extended to two-dimensional propagation, although with considerably greater analytical difficulty. The resulting rates should not be much different since the same physical principles are involved. So let us assume for the present that a typical time for these two waves to grow to appreciable steepness is of order 10–20 periods or greater. These two waves interact to produce a third wave of wave-number k_3 satisfying the resonance condition of § 2. The growth rate of this resonant wave is proportional to the steepness of the waves creating it, and the interaction time is of order 2–3 periods of the initial (k_1) waves, provided they are sufficiently steep. Since it was found that these resonant interactions are energy conserving, the energy of the k_3 wave must appear at the expense of the k_1 waves, tending to diminish their amplitudes. But the rate at which energy is drained from the k_1 waves is in turn proportional to their steepness, so in a short time an equilibrium steepness for the k_1 waves can appear, for which the drain of energy to the k_3 wave by the resonant interactions is precisely balanced by the input of energy to the k_1 wave from the wind. Further energy input from the wind now appears directly in the k_3 wave, causing it to grow until it breaks or finds some other wave with which to interact. Increasing the rate of energy input at wave-numbers k_1 simply increases the equilibrium steepness of the k_3 wave, up to a certain limit.

So we now see that energy input at wave-number k_1 can indeed produce a so called 'long crested' wave travelling with the wind, this being precisely the k_3 wave, which can grow to greater amplitude than the k_1 equilibrium amplitude. If we now calculate the phase speed of this k_3 wave according to the relation $C_3 = (g/k_3 + Tk_3)^{\frac{1}{2}}$, we find coincidentally that *this phase speed is exactly the same as the value of the critical convection velocity* U_{CR} *required to create the* k_1 *waves* shown in figure 9. Furthermore, the convection velocity of turbulent eddies of size k_3^{-1} is less than the velocity $C_3 = U_{CR}$, due to the shape of the mean velocity profile of the wind in the boundary layer above the water, so we have the following curious situation. Eddies of size k_3^{-1} are convected too slowly to account for straight crested k_3 waves according to the Phillips scheme, and eddies being convected at the correct speed required to create a long crested k_3 wave are of the wrong size, being of order k_1^{-1}. This seemingly paradoxical situation is precisely part of Miles's objection to Phillips resonant generation theory. But we have just demonstrated that a combination of the Phillips theory and the resonant interaction theory of this paper can indeed create a straight crested wave in this apparently wrong situation, thereby partially removing Miles's objections, at least for these types of configuration. So in this problem of wave generation by the wind the effects of these significant resonant interactions must also be included with the Phillips mechanism and the various Benjamin and Miles mechanisms, at least in the initial development stages for small wavelengths.

In terms of a random sea with appreciable large wave-number spectral content, use of this rough model suggests that through this interaction mechanism, energy appearing at some wave-number $k^{(1)}$ can rapidly appear at a higher wave-number $k^{(2)}$ say, and through further interaction between $k^{(2)}$ pairs, can appear at even higher wave-numbers $k^{(3)}$, say, until a wave-number around k^* is reached (see figure 8), at which point viscous dissipation predominates over the interaction mechanism. Wave-numbers for which this energy cascade through second-order resonant interaction is important are in the range $k_m/\sqrt{2} < k < k^*$. In terms of frequencies, $9 < f < 1400$ cycles per second. This is slightly greater than two decades in frequency, so we may infer that it is possible for the frequency spectra of surface elevations to achieve some sort of statistical equilibrium at large frequency through this cascading process, at least for some limited range of frequencies.† Furthermore, since the angles involved in these resonances are less than about $37\frac{1}{2}°$, we might expect the wave-number spectrum of surface elevations to exhibit strong directional characteristics at very high wave-numbers in a wind-blown sea. While it is not claimed here that the resonance mechanism is the predominant one at work, and certainly it is not the only one, it must be emphasized that in any investigation of the wave spectra at large wave-number or frequency, *the effects of these second-order resonant interactions must be included.*

It is possible to extend the type of resonant analysis presented here to third order simply by including higher-order terms in the two boundary conditions (3.10) and (3.11). The resonance condition will then involve quadruples of wave-numbers instead of triads. The algebra involved, however, would be prohibitive, and in addition, several other complications would arise. In considering the total energy, for instance, effects of the third-order terms must be included. Also, the phase speeds of the various interacting modes would depend on the amplitudes, the differences appearing due to the inclusion of the third-order terms.‡ This effect would tend to drive the resonances out of phase, destroying the interaction. However, based on the present analysis, the interaction times will be proportional to $(\hat{a}k)^{-2}$, which is considerably larger than the secondary interaction times of figure 8. So due to the weakness of these interactions, it is fairly safe to assume that they are insignificant for this range of large wave-numbers.

The results of this work can also shed light on some curious results of Wilton (1915) and Pierson & Fife (1961). They present a steady-state perturbation analysis for finite amplitude gravity-capillary waves propagating in one dimension, choosing for their frame of reference a co-ordinate system moving with the phase speed of the wave. With the elimination of time from their analysis, they find a singularity in this steady-state system for a single wave,

$$k_p = (g/2T)^{\frac{1}{2}} = k_m/\sqrt{2},$$

for which their perturbation scheme predicts infinite amplitudes. They eliminate this singularity by writing a different form for the equation of the free surface, including the second harmonic mode, namely $\eta = a\cos(g/2T)^{\frac{1}{2}}x + b\cos 2(g/2T)^{\frac{1}{2}}x$

† Note that for this kind of statistical equilibrium to occur, it is only necessary that the characteristic interaction times be smaller than the characteristic growth times due to the wind.

‡ See Pierson & Fife (1961), p. 167, equation (15).

(their notation). They then show that both components here travel at the same phase velocity, and to satisfy their requirement of steady-state periodic waves they show that $b = \pm\frac{1}{2}a$. But the second term on the right-hand side has wave-number $2k_p$, and since the phase velocities of both components are identical, the frequency of the second component is twice that of the first. These wave-numbers and frequencies satisfy the resonance condition given in the present work for the equal wave-number case, and in fact k_p is the smallest wave-number for which the equal wave-number resonance condition ($k_1 = k_2$) is satisfied. We identify both k_1 and k_2 with k_p, and k_3 with $2k_p$. The resonance angle β in this case is zero (see figure 2), or in other words, the propagation is in one dimension. Using the results (4.16) with $\hat{a}_2(\cos\beta)^{\frac{1}{2}} = \pm\frac{1}{2}\hat{a}_1$, the parameter of the elliptic functions is $\frac{1}{4}$, and the form of the free surface may be written as

$$\begin{aligned}\xi(x,t) = \hat{a}_1[&\mathrm{dn}\,(\tfrac{1}{2}(\hat{a}_1 k)\sigma_1(t-t_0)|\tfrac{1}{4})\\ &\pm \mathrm{cn}\,(\tfrac{1}{2}(\hat{a}_1 k_1)\,\sigma_1(t-t_0)|\tfrac{1}{4})]\cos(k_1 x - \sigma_1 t)\\ &\pm \tfrac{1}{2}\hat{a}_1\,\mathrm{sn}\,(\tfrac{1}{2}(\hat{a}_1 k_1)\,\sigma_1(t-t_0)|\tfrac{1}{4})\sin 2(k_1 x - \sigma_1 t).\end{aligned}$$

This is most easily interpreted as the interaction of a wave of wave-number $(g/2T)^{\frac{1}{2}}$ *with itself*. This implies that when a breakdown of steady-state analysis occurs, further analysis should include the time dependence of the phenomenon.

In conclusion, it has been shown through this new type of analysis that the second-order resonant interactions are significant. Any investigation of small-scale random water-wave phenomena must include the effects of these interactions. Evidently, detailed experimental investigations of the wind-wave problems and the various large wave-number spectra are urgently needed. Without more data, little more intelligent progress can be made.

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Appendix

From the combined boundary condition (3.10) we get

$$\begin{aligned}&\sum_{i=1}^{3}\left\{\left[\frac{\sigma_i}{k_i}(\ddot{a}_i - \sigma_i^2 a_i) + g\sigma_i a_i + Tk_i^2\sigma_i a_i\right]\sin\psi_i + \left(Tk_i^2 - 2\frac{\sigma_i^2}{k_i}\right)\dot{a}_i\cos\psi_i\right\}\\ &= \sum_{i=1}^{3}\{2a_i\dot{a}_i\sigma_i^2\cos\psi_i - [a_i\sigma_i(\ddot{a}_i - \sigma_i^2 a_i) + a_i^2\sigma_i k_i g - a_i^2 Tk_i^3\sigma_i]\sin\psi_i\}\cos\psi_i\\ &\quad + \sum_{\substack{j=1\\ j\neq k}}^{3}\sum_{k=1}^{3}\{[2a_j a_k(\sigma_j^2\sigma_k - \sigma_j\sigma_k^2\cos\theta_{jk}) + a_j a_k Tk_k^3\sigma_j\cos\theta_{jk}\\ &\quad - a_j\sigma_k(\ddot{a}_k - \sigma_k^2 a_k) - a_j a_k\sigma_k k_k g]\sin\psi_k\cos\psi_j\\ &\quad + 2a_j\dot{a}_k[(\sigma_k^2 - \sigma_j\sigma_k\cos\theta_{jk})\cos\psi_j\cos\psi_k - \sigma_j\sigma_k\sin\psi_j\sin\psi_k]\}, \qquad (\text{A } 1)\end{aligned}$$

remembering that $\cos\theta_{jk} = \cos\theta_{kj}$.

From the kinematic boundary condition (3.11) we get

$$\sum_{i=1}^{3} \dot{a}_i \cos\psi_i = 2\sum_{i=1}^{3} a_i^2 \sigma_i k_i \sin\psi_i \cos\psi_i + \sum_{\substack{j=1\\ j\neq k}}^{3}\sum_{k=1}^{3} a_j a_k(\sigma_j k_j + \sigma_k k_j \cos\theta_{jk}) \sin\psi_j \cos\psi_k. \quad \text{(A 2)}$$

Here, use has been made of the relation $\sigma_i^2 = gk_i + Tk_i^3$.

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